Chapter 3

Elements of Measurement, Representation, and Transformation Theory

3.1 Algebraic Preliminaries

3.1.1 - Metric Spaces

Consider an abstract set $M$, and mapping $g$:

$$M = \{X_n; \ n = 1, 2, 3, \ldots\} \equiv \{x, y, z, \ldots\} \quad g : M \times M \to \mathbb{R}$$

**Axiom:** $g$ has the properties:

1. $g(x, y) \geq 0 \ \forall x, y \in \mathbb{R} \quad g(x, y) = 0$ iff $x = y$
2. $g(x, y) = g(y, x) \ \forall x, y \in \mathbb{R}$ (Symmetry)
3. $g(x, z) \leq g(x, y) + g(y, z)$ (Triangle Inequality)

**Definition 1:** A set $M$ with a mapping $g$ subject to axioms (1)-(3) is called a *Metric Space* with *Metric* $g$

**Definition 2:** Consider an infinite sequence $X_n(n = 1, 2, \ldots) \in M$. Suppose $\exists x^* \in M$: $\lim_{n \to \infty} g(x^*, x_n) = 0$. Then $x^*$ is called the *limit of the sequence*.

Notation: $\lim_{n \to \infty} x_n = x^*$ or $x_n \Rightarrow x^*$

**Proposition 1:** A sequence can have at most one limit.

*Proof: See Problem # 48*
Proposition 2: Suppose $x_n \Rightarrow x^*$. Then
\[\forall \varepsilon > 0 \exists N \in \mathbb{N} : g(x_n, x_m) < \varepsilon \quad \forall n, m > N\]
Less formally:
\[g(x_n, x_m) \Rightarrow 0 \text{ for } n, m \to \infty\]

Definition 3: A sequence for which $g(x_n, x_m) \Rightarrow 0$ for $n, m \to \infty$ is called a Cauchy Sequence.

Remark: (1) Every sequence that has a limit is a Cauchy Sequence; but the converse is not true!

Examples:
1. $M = \mathbb{R}$, $g(x, y) = |x - y|$. Then every Cauchy sequence has a limit.
2. $M = \mathbb{Q}$ (Rationals), $g(x, y) = |x - y|$, $x_n = (1 + \frac{1}{n})^n$. This is a Cauchy sequence (proof Problem #...), but $\lim_{n \to \infty} x_n = e \notin \mathbb{Q}$

Definition 4: A metric space for which every Cauchy sequence has a limit is called Complete.

Proposition 3: A metric space that is not complete can always be made complete by adding a suitable set of points. The completion is unique up to Isomorphism. Proof: Books.

3.1.2 - Banach Spaces

Definition 1: An abstract set $V$ is called a Linear Space or Vector Space over $\mathbb{C}$ if it obeys the following axioms:

Axiom A:
1. $V$ is an abelian group under addition ("+") with zero element $\Theta$
2. The multiplication of elements of $V$ with numbers is defined such that:
   - $(ab)x = a(bx) \; \forall a, b \in \mathbb{C}, x \in V$ (Association)
   - $a(x + y) = ax + ay \; \forall a, b \in \mathbb{C}, x, y \in V$ (Distribution)
   - $1x = x$

Definition 2: A mapping $\|\| : V \to \mathbb{R}$ is called norm if it obeys:

Axiom B:
1. $\|x\| \geq 0 \; \forall x \in V$, $\|x\| = 0$ iff $x = \theta$
2. $\|x + y\| \leq \|x\| + \|y\|$ (Triangle inequality)
3. $\|ax\| = |a| \cdot \|x\| \; \forall a \in \mathbb{C}$ (Linearity)
Definition 3: The Distance $d$ between two elements of $V$ is defined as:

$$d(x, y) = \|x - y\| \quad \forall x, y \in V$$

Remark:
1. $d$ is a metric in the sense of ¶3.1.1
2. $\|x\| = d(x, \theta)$
3. Every linear space with a norm is in particular a metric space.

Definition 4: A linear space with a norm that is complete is called a Banach space, or B-space.

Definition 5: Let $B$ be a B-space over $C$. A Linear Functional $l$ in $B$ is a mapping $l : B \to C$ such that:

$$l(x + y) = l(x) + l(y)$$
$$l(ax) = al(x) \quad \forall x, y \in V, a \in C$$

Definition 6: The norm of a linear functional $l$ is defined as:

$$\|l(x)\| := \sup_{\|x\|=1} |l(x)|$$

Definition 7: Let $B$ be a B-space. The Dual Space or Conjugate Space of $B$, denoted by $B^*$, is defined as the space of all linear functionals in $B$.

Proposition:
1. $B^*$ is a linear space.
2. In $B^*$ a norm in the sense of def. 2 is defined by the norm of the functionals.
3. $B^*$ is complete, and therefore a B-space.

Proof: (1),(2) Problem 49, (3) Books

3.1.3 - Hilbert Space

Consider a linear space $H$ over $C$. Let there be a mapping $(\ , \) : H \times H \to C$ such that:

Axiom A:
1. $(x, y) = (y, x)^*$
2. $(x + y, z) = (x, z) + (y, z)$
3. \((x, x) \geq 0, (x, x) = 0 \) iff \(x = \theta\)

4. \((ax, y) = a^*(x, y)\)

**Definition 1:** The norm of \(x \in H\) is defined as \(\|x\| := (x, x)^{1/2}\)

**Definition 2:** The distance \(\rho\) between two elements \(x, y \in H\) is defined as

\[\rho(x, y) := \|x - y\|\]

and Cauchy sequences are defined as in §3.1.1 definition 3.

**Proposition 2:** \(\|\ldots\|\) is a norm in the sense of §3.1.2 Definition 2.

*Proof:* Problem # 49

**Axiom B:** \(H\) is complete, i.e., every Cauchy sequence has a limit.

**Remark:** 1. With these properties, \(H\) is in particular a B-space.

**Definition 3:** A space \(H\) with these properties is called a *Hilbert Space* or *H-Space*.

Now consider a fixed \(y \in H\) and define a linear functional \(l\)

\[l(x) := (y, x)\]

**Remark:**

2. This associates a linear functional \(l\) with every \(y \in H\)

2’ \(l\) is a mapping \(l : H \rightarrow \mathbb{C}\)

**Proposition 3:** Every linear functional in \(H\) can be uniquely written in this form.

*Proof:* Books

**Corollary:** The dual space \(H^*\) is isomorphic to \(H\).

**Definition 4:** Define a mapping \(<l> : H^* \times H \rightarrow \mathbb{C}\) by \(<l | x >= l(x)\)

**Remark:**

3. For every \(l \in H^* \exists y \in H : l(x) = (y, x) \implies <l | x >= (y, x)\)

4. \(H^*\) is isomorphic to \(H\) → We don’t have to distinguish between \(l\) and \(y\) and can write (sloppily) \(<y | x >= <l | x >= (y, x)\)

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3.2 The Axioms of Quantum Mechanics, and Elements of Measurement Theory

3.2.1 - Hilbert Space and Quantum Mechanics

Axiom 1: The states of a QM system correspond to the elements of an H-space $y$

Remarks:
1. See ¶1.3.1
2. States are denoted by $\Psi, |\Psi\rangle, |\Psi_n\rangle \equiv |n\rangle$
3. Elements of the dual space are denoted by $\langle \Psi |, \langle \Psi_n | \equiv \langle n |$
4. $\langle n |$ is a linear functional, not a vector! Its action on a state $m$ produces a number $\langle n | m \rangle \in \mathbb{C}$
5. $\langle n | m \rangle$ is a number; $|n\rangle\langle m|$ is an operator, i.e., $|n\rangle\langle m|m'| = \langle m|m'|n \rangle \propto |n\rangle$
6. If $|\Psi_1\rangle, |\Psi_2\rangle$ are states, then so is $\lambda_1|\Psi_1\rangle + \lambda_2|\Psi_2\rangle \forall \lambda_1, \lambda_2 \in \mathbb{C}$, i.e., the superposition principle is built into axiom 1.

3.2.2 - Projectors and Probability

Definition 1:
(a) A linear operator $\hat{P}$ defined on $y$ is called a projector if $\hat{P}^\dagger = \hat{P}$ and $\hat{P}^2 = \hat{P}$
(b) $U_p := \hat{P}y$ is called the subspace onto which $\hat{P}$ projects.
$U_p^\dagger := (1 - \hat{P})y$ is called the orthogonal complement of $U_p$

Remark:
1. Let $\hat{P}$ be a projector and $U_p \subseteq y$ the corresponding subspace of $y$. Then every vector $\Psi \in y$ can be uniquely written as $\Psi = \Psi_1 + \Psi_2$ with $\Psi_1 \in U_p, \Psi_2 \in U_p^\perp$
then $\hat{P}\Psi_1 = \Psi_1, \hat{P}\Psi_2 = 0$
$\hat{P}\Psi = \Psi_1, (1 - \hat{P})\Psi = \Psi_2$

2. Let $U_p$ be 1-d with basis $|e\rangle$. Then

$$\hat{P}|\Psi\rangle = \frac{|e\rangle\langle e|\Psi\rangle}{\langle e|e\rangle} \sim \hat{P} = \frac{|e\rangle\langle e|}{\langle e|e\rangle}$$

2'. Let $\{|e_n\rangle, n = 0, 1, ...\}$ be an orthonormal basis of $U_p$. Then
\[ \hat{P} = \sum_n |e_n\rangle\langle e_n| \quad \text{and} \quad \hat{P}|\Psi\rangle = \sum_n \langle e_n|\Psi\rangle|e_n\rangle \]

3. Let \( U_p = y \). Then \( \hat{P} = \sum_n |e_n\rangle\langle e_n| = \hat{1} \) See §1.3.3 remark 6

Axiom 2:

(a) The properties of a QM system correspond to a projector \( \hat{P} \) (or to the corresponding subspace \( U_p \)).

(b) The probability of a QM system in state \( |\Psi\rangle \) to have the property \( P \) is

\[ w_{\Psi}(P) = \frac{\langle \Psi|\hat{P}|\Psi\rangle}{\langle \Psi|\Psi\rangle} \]

(c) If the system is in state \( |\Psi\rangle \), and \( P \) is measured, then after the measurement the system is in the state \( \hat{P}|\Psi\rangle \).

Remark

4. Axiom 2 (c) is often referred to as "collapse of the wave function." We also say "the measurement reduces \( |\Psi\rangle \) to \( \hat{P}|\Psi\rangle \)."

5. "Property" is to be understood as a question that can only be answered by either "yes" or "no". Examples: "Particle is in volume \( V \)"; "Particle has spin up", etc. If we say "\( P \) is measured" we mean "the question was asked: 'Does the system have the property \( P \)?', and the answer was affirmative.

6. \( w_{\Psi}(P) = \frac{\langle \Psi|\hat{P}|\Psi\rangle}{\langle \Psi|\Psi\rangle} \geq 0 \)

\[ w_{\Psi}(P) \geq 0 \]

Remark (1) \( \Rightarrow \Psi = \Psi_1 + \Psi_2 \) with \( \Psi_1 \in U_p, \Psi_2 \in U_p^\perp \) and \( \hat{P}\Psi = \Psi_1 \)

\[ w_{\Psi}(P) = \frac{\langle \Psi_1 + \Psi_2|\Psi_1\rangle}{\langle \Psi|\Psi\rangle} = \frac{||\Psi_1||^2}{||\Psi_1||^2 + ||\Psi_2||^2} \leq 1 \]

\[ 0 \leq w_{\Psi}(P) \leq 1 \] So it can be interpreted as a probability.

Examples:

1. Property: "Particle is in volume \( V \)". \( \hat{P} \) is defined by:

\[ \hat{P}\Psi(x) = \begin{cases} \Psi(x) & \text{if } x \in V \\ 0 & \text{otherwise} \end{cases} \]

\( \hat{P} = \hat{P}^\dagger \) and \( \hat{P}^2 = \hat{P} \Rightarrow \hat{P} \) is a projector.

\[ w_{\Psi}(P) = \frac{\langle \Psi|\hat{P}\Psi\rangle}{\langle \Psi|\Psi\rangle} = \frac{\int d\vec{x}|\Psi(\vec{x})|^2}{\int d\vec{x}|\Psi(\vec{x})|^2} \]

This is consistent with §1.2.3
2. Let \( \{ \Psi_n \} \) be a complete orthonormal set. Let a particle be in state \( \Psi = \sum_n c_n \Psi_n \)

Let \( \Psi_m \) be the property: "particle is in state \( \Psi_m \)"

projector: \( \hat{P} = |\Psi_m\rangle \langle \Psi_m| \)

\[
w_\psi(P) = \frac{\langle \Psi | \hat{P} | \Psi \rangle}{\langle \Psi | \Psi \rangle} = \frac{|\sum_n c_n^* \langle \Psi_n | \Psi_m \rangle|^2}{\langle \Psi | \Psi \rangle^2} = \frac{|c_m|^2}{\sum_n |c_n|^2}
\]

3.2.3 - Multiple Measurements

**Problem:** Axiom 2(c) \( \rightarrow \) measurement changes the state of the system! Need to check consistency!

**Repeated measurement:**

Measure P: \( \Psi \rightarrow \hat{P} \Psi \)

Measure P again: \( \hat{P} \hat{P} \Psi = \hat{P}^2 \Psi = \hat{P} \Psi \) No further change!

\[
w_{\hat{P} \Psi}(P) = \frac{\langle \hat{P} \Psi | \hat{P} \Psi \rangle}{\langle \hat{P} \Psi | \hat{P} \Psi \rangle} = \frac{\langle \hat{P} \Psi | \hat{P} \hat{P} \Psi \rangle}{\langle \hat{P} \Psi | \hat{P} \Psi \rangle} = \frac{\langle \hat{P} \Psi | \hat{P} \Psi \rangle}{\langle \hat{P} \Psi | \hat{P} \Psi \rangle} = 1
\]

\( \sim \) Measurements are reproducible!

**Successive Measurements:** measure P, then Q, with checks

\[
\begin{align*}
\Psi & \xrightarrow{1\text{st}} \hat{P} \Psi & \xrightarrow{2\text{nd}} \hat{Q} \Psi \\
& \downarrow \text{1st Check} & \downarrow \text{2nd Check} \\
w_1(P) = 1 & w(Q) = 1 & w_2(P) = ??
\end{align*}
\]

\[
w_2(P) = \frac{\langle \hat{Q} \hat{P} \Psi | \hat{P} \hat{Q} \hat{P} \Psi \rangle}{\langle \hat{Q} \hat{P} \Psi | \hat{Q} \hat{P} \Psi \rangle} = \frac{\langle \hat{Q} \hat{P} \Psi | \hat{P}^2 \hat{Q} \hat{P} \Psi \rangle}{\langle \hat{Q} \hat{P} \Psi | \hat{Q} \hat{P} \Psi \rangle} = \frac{\| \hat{P} \hat{Q} \hat{P} \Psi \|^2}{\| \hat{P} \hat{Q} \hat{P} \Psi \|^2} \leq 1
\]

\( \sim \) in general \( w_2 < 1 \) \( \sim \) in general, measurement of Q destroys reproducibility of the previous measurement of P.

**Remark:** (1) Any interaction of the "system" with the rest of the world (including the observer!) is a "measurement" \( \sim \) deep problem!

**Exception:** Let \( [\hat{Q}, \hat{P}] = 0 \) \( \sim \) \( \hat{P} \hat{Q} \hat{P} = \hat{Q} \hat{P}^2 = \hat{Q} \hat{P} \sim w_2(P) = 1 \)

**Definition:** Two Properties \( P_1, P_2 \) with \( [\hat{P}_1, \hat{P}_2] = 0 \) are called *simultaneously measurable.*
Remark: (2) This sheds new light on § 1.2.6

Examples: (1) $P_{1,2} = \text{particle in volume } V_{1,2}$

$$\hat{P}_1 \hat{P}_2 \psi(x) = \begin{cases} \psi(x) & \text{if } x \in V_1 \cap V_2 \\ 0 & \text{otherwise} \end{cases} = \hat{P}_2 \hat{P}_1 \psi(x) \sim [\hat{P}_1, \hat{P}_2] = 0$$

Remark: (3) § 1.2.7 $\rightsquigarrow$ Observables correspond to general hermitian operators. Connection with properties and projectors? Answer: Reduce measurements to a succession of yes/no decisions (e.g., determine position by bisection). Difficult to make rigorous.
3.3 Elements of Representation Theory

3.3.1 - Representation of states

Question: How is the wave function $\psi(x)$ related to the abstract state $|\psi\rangle \in \mathbb{y}$ of §3.2.1 axiom 1?

Remarks:

1. $\psi_a(x) \in \mathbb{C}$ (Not $\in \mathbb{y}$!)
2. $a$ characterizes the state ('state index')
3. $|\psi_a\rangle \equiv |a\rangle \in \mathbb{y}$
4. States in dual space: $|a\rangle^\dagger \equiv \langle a| \in \mathbb{y}^*$
5. $\langle a|$ is a functional that's isomorphic to an element of $\mathbb{y}$.
6. $\langle a|b\rangle \in \mathbb{C}$ is the result of the functional $\langle a|$ acting on the state $|b\rangle$, or, equivalently, the scalar product of the state $|b\rangle$ with the state that's isomorphic to $\langle a|$

Linear algebra $\Rightarrow$ we know how to represent an abstract state $|a\rangle$: Choose a basis $\{|a_n\rangle, n = 0, 1, ...\}$ in $\mathbb{y}$, and expand $|a\rangle$:

$$|a\rangle = \sum_{n=0}^{\infty} a_n |e_n\rangle \text{ with } a_n = \langle e_n|a\rangle$$

Remarks: (7) The ”vector” $(a_0, a_1, ...)$ is a representation of $|a\rangle$. It depends on the basis alone.

(8) §1.3.3 $\Rightarrow$ The eigenvectors of any hermetic operator in $\mathbb{y}$ form a basis.

Definition 1: Let $|a\rangle \in \mathbb{y}$ be a state, and let $\hat{F} = \hat{F}^\dagger$ be a hermetic operator with eigenvectors $|f_n\rangle$ and eigenvalues $f_n : \hat{F}|f_n\rangle = f_n|f_n\rangle$.

Then the coefficients $\psi_a(f_n)$ in the expansion

$$|a\rangle = \sum_n \psi_a(f_n)|f_n\rangle$$

are called the $F$-representation of the state $|a\rangle$

Remark:

(9) The spectrum $\{f_n\}$ may be discrete or continuous. Replace $\sum_n \rightarrow \int \delta f$ as appropriate.

(10) §1.2.5 $\Rightarrow \psi_a(f_n) = \langle f_n|a\rangle, \quad \psi_a(f_n)^* = \langle a|f_n\rangle$
Example:

\[
\hat{F} = \hat{x}
\]

**Eigenvalues:** \(x\) (spectrum continuous)

**Orthonormality:** \(\langle x|y \rangle = \delta(x - y)\)

**Completeness:** \(\int dx |x\rangle \langle x| = \mathbb{1}\)

\[\psi_a(x) = \langle x|a \rangle\]

Coordinate representation of \(|a\rangle \equiv |\psi_a\rangle\), a.k.a. wavefunction.

Projector for the property "Particle at position \(x\): \(\hat{P} = |x\rangle \langle x|\)

Probability density for particle in state \(|a\rangle\) to be at point \(x\):

\[
\rho_a(x) = \frac{\langle a|\hat{P}|a \rangle}{\langle a|a \rangle} = \frac{\langle a|x\rangle \langle x|a \rangle}{\langle a|a \rangle} = \frac{|\psi_a(x)|^2}{\int dx |a| \langle x|a \rangle} = \int dx |\psi_a(x)|^2
\]

Remark:

(11) The postulate of 1.2.3 has now been derived from the more general axiom 2 (b) in 3.2.2

### 3.3.2 - Change of Representation

Let \(\hat{F}, \hat{G}\) be hermitian operators with eigenfunctions \(|f_n\rangle, |g_n\rangle\) (\(n = 0, 1, 2,\ldots\))

\(\{f_n\}\) and \(\{g_n\}\) provide two basis sets in \(y\)

Completeness implies:

\[
\sum_n |g_n\rangle \langle g_n| = \mathbb{1}.
\]

\(\leadsto\) Two representations of a state \(|a\rangle\) are given by:

\[
|a\rangle = \sum_n \psi_a(f_n) |f_n\rangle \quad \psi_a(f_n) = \langle f_n|a \rangle
\]

\[
= \sum_n \psi_a(g_n) |g_n\rangle \quad \text{with} \quad \psi_a(g_n) = \langle g_n|a \rangle
\]

We can transform one into the other by (*)

\[
\psi_a(f_n) = \langle f_n|a \rangle = \sum_m \langle f_n|g_m \rangle \langle g_m|a \rangle = \sum_m \langle f_n|g_m \rangle \psi_a(g_m)
\]

Definition: A matrix \(M\) is called unitary if \(M^+ = M^{-1}\), with \((M^+)^{nm} = M_{nm}^*\) the hermitian conjugate.

Theorem: The transformation between the F-representation and the G-representation of the state \(|a\rangle\) is accomplished by the unitary matrix \(S_{nm} = \langle f_n|g_m \rangle\)
Proof:

\[(+) \Rightarrow S_{nm} \text{ accomplishes the transformation.}\]
\[
(SS^+)_{nm} = \sum_n S_{nm} S_{mn}^* = \sum_n (f_n | g_n \rangle \langle f_m | g_m \rangle^*)
= \sum_n (f_n | g_n \rangle \langle g_n | f_m \rangle = (f_n | f_m \rangle = \delta_{mn}
\Rightarrow S \text{ is unitary} \quad \square
\]

Example:

(1) Momentum Representation:
Let \( \hat{G} = \hat{x} \) and \( \hat{F} = \hat{p} \)

Eigenvectors: \( |p\rangle \)
Eigenvalues: \( p \) (continuous spectrum)
Orthogonality: \( \langle p | k \rangle = \delta(p - k) \)
Completeness: \( \int dp |p\rangle \langle p| = 1 \)

\( \Rightarrow \psi_a(p) = \langle p |a \rangle = \int dx |p \rangle \langle x |a \rangle = \int dx |p \rangle \psi_a(x) \)

But \( \langle x |p \rangle^* = x\text{-representation of } p\text{-eigenstates.} \)
\( \psi_p(x)^* = \frac{1}{\sqrt{2\pi\hbar}} e^{-i(p/\hbar)x} \) (¶1.3.4)

\( \Rightarrow \psi_a(p) = \int dx \frac{1}{\sqrt{2\pi\hbar}} e^{-i(p/\hbar)x} \psi_a(x) \)

Conclusion: The p-representation of \( |a\rangle \) is the Fourier transform of the x-representation of \( |a\rangle \)

Projector for the property "particle has momentum p": \( \hat{P} = |p\rangle \langle p| \)

\( \Rightarrow \) Probability density for particle in state \( |a\rangle \) to have momentum p:
\[
\rho_a(p) = \frac{|a \rangle \langle p|}{\langle a |a \rangle} = \frac{|\langle a |p \rangle|^2}{\int dp |a \rangle \langle p|} = \frac{|\psi_a(p)|^2}{\int dp |\psi_a(p)|^2}
\]

Remark: (1) ¶1.2.6 agrees, but the current formulation is much more general!

Example:

(2) Energy Representation:
Let \( \hat{F} = \hat{H} \)

Eigenvectors: \( |n\rangle \)
Eigenvalues: \( E_n \) (Spectrum discrete or continuous or both)
Orthogonality: \( \langle n |m \rangle = \delta_{mn} \)
Completeness: \( \sum_n |p \rangle \langle p| = 1 \)
\[ \psi_a(E_n) = \langle n|a \rangle = \int dx \langle n|x \rangle \langle x|a \rangle = \int dx \langle n|x \rangle \psi_a(x) \]

With \( \langle n|x \rangle = \langle x|n \rangle^* = \psi_n(x)^* = x\)-representation of the energy eigenstates.

\[ \psi_a(E_n) = \int dx \, \psi_n(x)^* \psi_a(x) \]

Projector for property "particle has energy \( E_n \): \( \hat{P} = |n\rangle \langle n| \)

\[ w_0(E_n) = \frac{|\langle a|n \rangle|^2}{\langle a|a \rangle} = \frac{|\psi_a(E_n)|^2}{\sum_n |\psi_a(E_n)|^2} \]

Remarks:

(2) §1.2.2 example 2

(3) Continuous Spectrum \( \leadsto \) probability density \( \rho \)

Discrete Spectrum \( \leadsto \) probability \( w \)

3.3.3 - Representation of Operators

Let \( \hat{F} \) be an operator in \( y \) defined by its action on all possible \( |a \rangle \in y \):

\[ \hat{F}|a \rangle = |b \rangle \]

§3.3.1,3.3.2 \( \leadsto \) The representations of \( |b \rangle \) give rise to representations of \( \hat{F} \).

**x-representation:** \( \langle x|b \rangle = \langle x|\hat{F}|a \rangle = \int dy \, \langle x|\hat{F}|y \rangle \langle y|a \rangle =: \int dy \, F_{xy}(y|a) \) with \( F_{xy} := \langle x|\hat{F}|y \rangle \)

**p-representation:** \( \langle p|b \rangle = \int dp' \, F_{pp'}(p'|a) \) with \( F_{pp'} = \langle p'|\hat{F}|p \rangle \)

**E-representation:** \( \langle n|b \rangle = \sum_m F_{nm}(m|a) \) with \( F_{mn} = \langle n|\hat{F}|m \rangle \)

**Definition:** Let \( \hat{G} \) be a hermitian operator whose eigenfunctions \( |g_n \rangle \) provide a basis in \( y \). The set

\[ \{F^G_{mn}\} = \{\langle g_n|\hat{F}|g_m \rangle\} \]

is called the \( G\)-representation of the operator \( \hat{F} \).

**Theorem:** The transformation from the \( F\)-representation of an operator \( \hat{O} \) to the \( G\)-representation is given by the same unity matrix \( S_{nm} = \langle f_n|g_m \rangle \) that transforms the representations of the states, viz.

\[ O^G_{nm} = (SOG S^{-1})_{nm} \]
Proof: \[ O_{nm}^F = \langle f_n | \hat{O} | f_m \rangle = \sum_{nm} \langle f_n | g_n \rangle \langle g_n | \hat{O} | g_m \rangle \langle g_m | f_m \rangle \]
\[ = \sum_{nm} S_{nm} O_{nm}^G \langle f_m | g_m \rangle^* = \sum_{nm} S_{nm} O_{nm}^G S^{-1}_{mn} \]
\[ = (SO^G S^{-1})_{nm} \]

Example: (1) **Position Operator:** \( \hat{x} |x\rangle = x |x\rangle \)

Coordinate Representation: \( \langle y | \hat{x} | x \rangle = x \langle y | x \rangle = x \delta (y-x) \)

Momentum Representation:

\[
\langle p | \hat{X} | p' \rangle = \int dp \int dp' \frac{1}{\sqrt{2\pi\hbar}} e^{-i(p'/\hbar)p} \frac{1}{\sqrt{2\pi\hbar}} e^{-i(p-\hbar)p'/\hbar} \psi_a(p', \hbar) \]
\[ = \frac{i}{\hbar} \frac{\partial}{\partial p} \delta (p-p') \]

Remark: (1) let \( |a\rangle \equiv |\psi_a\rangle \) be an arbitrary state in \( \mathcal{H} \), then:

\[
\langle x | \hat{x} | a \rangle = \int dydz \langle x | y \rangle \langle y | \hat{x} | z \rangle \langle z | a \rangle = \int dydz \delta (x-y) y \delta (y-z) \psi_a(z) = x \psi_a(x) \quad \text{Agrees with Ch 1} \]

\[
\langle p | \hat{x} | a \rangle = \int dp dp' \langle p | \hat{p} | p' \rangle \langle p' | \hat{x} | p \rangle \langle p | a \rangle = \int dp dp' \delta (p-p') i\hbar \frac{\partial}{\partial p} \delta (p^2 - p^2) \psi_a(p) \]
\[ = i\hbar \frac{\partial}{\partial p} \psi_a(p) \]

Example: (2) **Momentum Operator:** \( \hat{p} |p\rangle = p |p\rangle \)

Momentum Representation: \( \langle p' | \hat{p} | p \rangle = p \delta (p' - p) \quad \Rightarrow \quad (\hat{p})_{pp'} = p \delta (p - p') \)
Coordinate Representation:

\[
(\hat{p})_{xy}^x = \int dp dp' \langle x|p\rangle \langle p|\hat{p}|p'\rangle \langle p'|y\rangle
= \frac{1}{2\pi \hbar} \int dp dp' e^{ipx/\hbar} \delta(p-p') e^{ip'y/\hbar}
= \frac{1}{2\pi \hbar} \int dp pe^{ip(x-y)/\hbar} = -i\hbar \frac{\partial}{\partial x} \delta(x-y)
\]

Remark: (2) Let \( |a\rangle \equiv |\psi_a\rangle \in y \). Then

\[
\langle x|\hat{p}|a\rangle = \int dy dz \langle x|y\rangle \langle y|\hat{p}|z\rangle \langle z|a\rangle
= \int dy dz \delta(x-y)(-i\hbar) \frac{\partial}{\partial y} \delta(y-z)\psi_a(z)
= -i\hbar \int dz \frac{\partial}{\partial x} \delta(x-z)\psi_a(z)
= -i\hbar \frac{\partial}{\partial x} \psi_a(x)
\]

Agrees with Ch. 1

Example: (3) Hamiltonian Operator: \( \hat{H}|n\rangle = E_n|n\rangle \)

Energy Representation: \( \langle m|\hat{H}|n\rangle = E_n \delta_{nm} \leftrightarrow (\hat{H}E)_{nm} = \delta_{nm}E_n \)

Coordinate Representation:

\[
(\hat{H}^x)_{xy} = \sum_{n,m} \langle x|n\rangle \delta_{nm} E_n \langle m|y\rangle
= \sum_n \psi_n(x) E_n \psi_n^*(x)
\]

Remark: (3) All concepts and results from Ch 1 emerge as special cases of this more general formalism.